Complete Fourth Power Exponential Distribution Geometric and Statistical study

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ABSTRACT

In this paper, the interest is focussed on the geometric and statistical properties of the perfect 4^{th} power exponential distribution (FPED) termed complete FPED. The form of the distribution is a special form of the 4^{th} power exponential distribution in the general form. Comparison between this complete 4^{th} power exponential distribution and the normal distribution is presented geometrically, graphically and statistically for different values of the parameters.

Key Words

4^{th} power exponential distribution - perfect 4^{th} power exponential distribution.

1. Introduction :

The Fourth Power Exponential Distribution with general form [1] is given by :

$$f(x) = K_1 \, e^{-a(x^4 + bx^3 + cx^2 + dx)} \qquad (1\text{-}1)$$

where $a > 0$ and $-\infty < x < \infty$

A family of forms of this distribution depending upon the values chosen for b, c and d could be derived. For example, the distribution defined by:

$$f_{a,\alpha}(x) = K \, e^{-a(x-\alpha)^4} \qquad -\infty < x < \infty,\ -\infty < \alpha < \infty \text{ and } a > 0 \qquad (1\text{-}2)$$

is a member of that family where

$b = -4\alpha$, $c = 6\,\alpha^2$, $d = -4\,\alpha^3$, $k_1 = k\,e^{-a\alpha^4}$

This form could be termd as "Complete Fourth Power. Exponential Distribution " (CFPED).

In this paper the geometric and statistical criteria of this CFPED are studied together with comparison with the normal distribution.

2. The complete Fourth Power Exponential Distribution:

Let x have the probability density function (pdf):

$$f_{a,\alpha}(x) = k\ e^{-a(x-\alpha)^4} \tag{2-1}$$

where $-\infty < x < \infty, -\infty < \alpha < \infty$ and $a > 0$

This distribution is unimodal and symmetric about its mean α. The central moments of this variate about α are :

$$\mu_{2r+1} = 0 \qquad \text{for } r = 0,1,2,3,\ldots \tag{2-2}$$

and $$\mu_{2r} = k\ \Gamma\left(\frac{(2r+1)}{4}\right) / 2\ a^{\frac{2r+1}{4}} \qquad \text{for } r = 0,1,2,\ldots \tag{2-3}$$

Now, letting $r = 0$ in (2-3) gives :

$$k = 2\ a^{\frac{1}{4}} / \Gamma(\tfrac{1}{4}) \tag{2-4}$$

Substituting the value of k given by (2-4) in (2-1) and (2-3) , we get

$$f_{a,\alpha}(x) = \left((2a^{\frac{1}{4}}) / \Gamma(\tfrac{1}{4})\right) e^{-a(x-\alpha)^4} \tag{2-5}$$

where $a > 0$, $-\infty < x < \infty$ and $-\infty < \alpha < \infty$
and

$$\mu_{2r} = \Gamma(\frac{2r+1}{4} / a^{\frac{r}{2}}\ \Gamma(\tfrac{1}{4}) \qquad r = 1,2,3,\ldots \tag{2-6}$$

Consequentely, the coefficient of curtosis will be :

$$B = \mu_4/\mu_2^2 = [\Gamma(\tfrac{1}{4})\ /2\ \Gamma(3/4)]^2 = 2.1882494 \tag{2-7}$$

which is independent of a.

Putting $\frac{d^2f(x)}{dx^2} = 0$, from which the real points of inflection of this distribution are at :

$$x = \alpha \pm (3/4a)^{\frac{1}{4}} \qquad (2\text{-}8)$$

3- Comparison between the Complete Fourth Power Exponential Distribution with $\alpha = 0$ and the Standard Normal Distribution.

If $\alpha = 0$, then the probability density function of the CFPED will be :

$$f_{a,0}(x) = (2a^{\frac{1}{4}} / \Gamma(\tfrac{1}{4}))\ e^{-ax^4} \qquad -\infty < x < \infty \text{ and } a > 0 \qquad (3\text{-}1)$$

Now, let Z be N(0,1), then the probability density function of Z is given by :

$$g(z) = (1/\sqrt{2\pi})\ e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty \qquad (3\text{-}2)$$

The centeral moments for the standard normal distribution [2] are given by :

$$\mu^*_{2r+1} = 0 \qquad r = 0, 1, 2, \ldots \qquad (3\text{-}3)$$

$$\mu^*_{2r} = 2^r\ \Gamma\left(\frac{2r+1}{2}\right)/\Gamma(\tfrac{1}{2}) \qquad r = 1, 2, 3, \ldots \qquad (3\text{-}4)$$

and the coefficient of curtosis is :

$$B^* = 3 \qquad (3\text{-}5)$$

Also, the points of inflection are at

$$x^* = \pm 1 \tag{3-6}$$

Comparing between the CFPED when $\alpha = 0$ and the standard normal distribution, the following results are obtained noting that parameters of normal are distinguished by (*) and those corresponding to CFPED are without (*) :

Result 1:

From (2-6) and (3-3), the following is true for the moments :

$$\mu_{2r} \lesseqgtr \mu^*_{2r} \text{ when a } \gtreqless \left(\Gamma(\tfrac{1}{2}) \quad \Gamma(\tfrac{2r+1}{4}) / \; 2^r \; \Gamma(\tfrac{1}{4}) \quad \Gamma(\tfrac{2r+1}{2})\right)^2$$

$$r = 1,2,3,\ldots \tag{3-7}$$

For example,

$$\mu_2 \lesseqgtr \mu^*_2 \quad \text{when a} \gtreqless 0.1142465 \tag{3-8}$$

and

$$\mu_4 \lesseqgtr \mu^*_4 \quad \text{when a} \gtreqless 0.08\bar{3} \tag{3-9}$$

Result 2 :

From (2.7) and (3-4), the following is true for the curtosis :

$$B = 2.1882494 < 3 = B^* \qquad \text{for all a} \tag{3-10}$$

i.e. The complete fourth power exponential distribution is more flat around its center (0) than the standard normal distribution.

Result 3:

From (2-8) and (3-5), the following is true for the points of inflection:

$$| x | = (\frac{3}{4a})^{\frac{1}{4}} \gtreqless | x^* | = 1 \qquad \text{when } a \gtreqless \frac{3}{4} \tag{3-11}$$

The following table summarizes the results about μ_2 , μ_4 and the points of inflection according to different values of a:

Table (1)
Relations between 2^{nd} moments, 4^{th} moments
and points of inflection of $f_{a,0}$ (x) and g (x)
for different values of a.

CFPED Parameter \ a	$0 < a \leq 0.083$	$.083 < a \leq .114$	$0.114 < a < 0.75$	$a \geq 0.75$
μ_2	$> \mu_2^*$	$\geq \mu_2^*$	$< \mu_2^*$	$< \mu_2^*$
μ_4	$\geq \mu_4^*$	$< \mu_4^*$	$< \mu_4^*$	$< \mu_4^*$
$\lvert x \rvert$	$> \lvert x^* \rvert$	$< \lvert x^* \rvert$	$< \lvert x^* \rvert$	$\geq \lvert x^* \rvert$

Result 4 :

The following shows comparison between pdf's of CFPD with $\alpha = 0$ and standard normal distribution at x = 0, i.e. $f_{a,0}(0)$ and g(0).

Assuming $f_{a,0}(0) = \lambda\, g(0)$

then $(2\, a^{\frac{1}{4}}/\Gamma(\frac{1}{4})\ ) = \lambda\ (1/\sqrt{2\Pi})$

from which

$$\lambda = \frac{2^{\frac{3}{2}}\, a^{\frac{1}{4}}\, \sqrt{\Pi}}{\Gamma(\frac{1}{4})} \tag{3-12}$$

which depends on a.

Result 5:

The points of intersection of the two curves representing the pdfs of CFPED and the standard normal distribution are given by:

$$x = \pm \left(\sqrt{\frac{1}{4a}}\right) \sqrt{1+\sqrt{1+16a\ \ln(k\sqrt{2\pi})}} \qquad (3\text{-}13)$$

<u>Special Cases:</u>

<u>Case (1):</u>

When the 2^{nd} central moment of the CFPED is taken equal to the 2^{nd} central moment of the standardized normal. i.e. $\mu_2 = \mu_2^* = 1$
Then from (2-6),

$$a = \left(\frac{\Gamma(\frac{3}{4})}{\Gamma\frac{1}{4}}\right)^2 = 0.114236635 \qquad (3\text{-}14)$$

Hence, the CFPED density given by (3-1) will be:

$$f_1(x) = 0.320701844\ \bar{e}^{0.114236635\ x^4} \qquad (3\text{-}15)$$

The graphs of f_1 and g are given in figure (1).

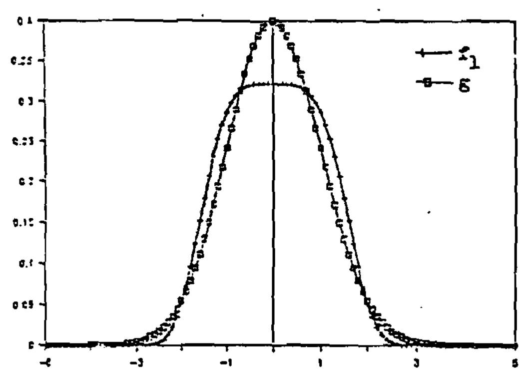

Figure (1)

Now, substituting the value of a given by (3-14) into (2-6),(2-8),(3-12) and (3-13), the following phenomena are obtained:

(i) The even central moments:

$$\mu_{2r} = \frac{\Gamma(\frac{2r+1}{4})}{(0.114236635)^{\frac{r}{2}}\ \Gamma(\frac{1}{4})} \qquad r=1,2,3,\ldots \qquad (3\text{-}16)$$

In particular,

$$\mu_4 = 2.1884404 \qquad (3.17)$$

(ii) The points of inflection:

$$x = \pm\ 1.600714 \qquad (3.18)$$

i.e. $|x| > |x^*| = |$ for the standard normal distribution.

(iii) $\lambda = 0.8038556$

Consequentely,

$$f_1(0) = 0.8038556\ g(0) \qquad (3\text{-}19)$$

i.e. The peak of the CFPED is below the peak of the standard normal distribution.

(iv) There are four points of intersection between the pdf curves f_1 and g given by:

$$x = \pm\ 0.7013491\ ,\ \ \pm\ 1.9710379 \qquad (3\text{-}20)$$

(v) The following table gives areas under both $f_{1(x)}$ and $g(x)$ at different values of x [i.e $\int_{-x}^{x} f_1(x)dx$ and $\int_{-x}^{x} g(x)dx$]:

Table (2)

Some areas under the curves of

$f_1(x)$ and $g(x)$ from $-x$ to x

x		Area under	
Description	Value	$f_1(x)$	$g(x)$
1st point of intersection between f_1 & g	0.7013491	0.4465385	0.5160
Point of inflection of g	1.0	0.627202562	0.6826
Point of inflection of f_1	1.6007149	0.89972215	0.8904
Arbitrary point	1.673068	0.920377348	0.905
Arbitrary point	1.80406188	0.95	0.9282
2^{nd} point of intersection between f_1 & g	1.9710379	0.97513252	0.9512
Arbitrary point	2.0	0.97839964	0.9544
Arbitrary point	2.14830	0.99	0.9684

The previous table shows that the area under the CFPED (f_1) is less than that under the standard normal distribution for values of x less than the point of inflection of the CFPED and then the area under f_1 starts to be larger than that under g. This is due to the fact that the curve of the CFPED is more flatter than the normal curve.

Figure (1) illustrates most of the previous criteria.

Case (2):

When the two peaks of the CFPED and the standard normal distribution are equal.

i.e. $f_{a,0}(0) = g(0)$

In this case,

$$\frac{2(a)^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} = \frac{1}{\Gamma(2\pi)} \tag{3.21}$$

from which

$$a = \frac{1}{64}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})}\right)^4 = 0.27335851 \tag{3-22}$$

Hence, the complete 4^{th} power exponential density given by(3-1) will be:

$$f_2(x) = 0.3989544\ \bar{e}^{0.2735851\ x^4} \qquad -\infty < x < \infty \tag{3-23}$$

The graphs of f_2 and g are given in figure (2).

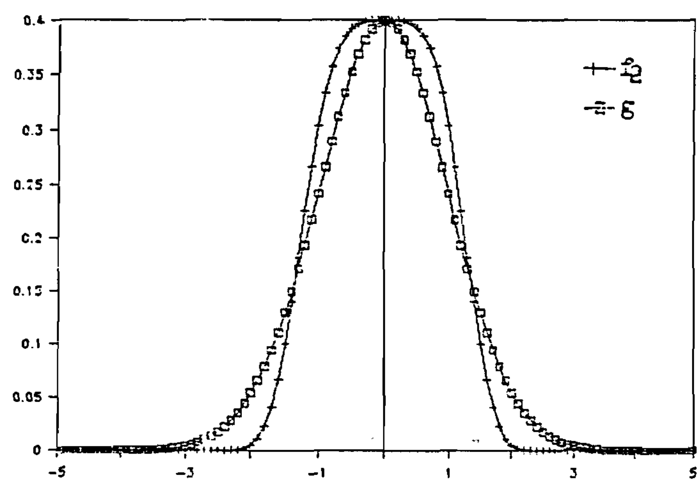

Figure (2)

Now, substituting the value of a given by (3-22) into (2-6),(2-8) and (3-13), the following phenomena are obtained:

(i) The even central moments:

$$\mu_{2r} = \frac{\Gamma(\frac{2r+1}{4})}{(0.2735851)^{\frac{r}{2}}\ \Gamma(\frac{1}{4})} \qquad r = 1,2,3,\ldots \qquad (3\text{-}24)$$

In particular,

$$\mu_2 = 0.8481173 \quad \text{and} \quad \mu_4 = 0.9137927 \qquad (3\text{-}25)$$

(ii) The points of inflection are:

$$x = \pm\ 1.2867441 \qquad (3\text{-}26)$$

i.e. $|x| > |x^*| = 1$ for the standard normal distribution

(iii) The points of intersection:

There are three points of intersection given by

$$x=0,\ \pm\ 1.351882 \qquad (3\text{-}27)$$

(iv) The following table gives areas under both $f_2(x)$ and $g(x)$ at different values of x (i.e. $\int_{-x}^{x} f_2(x)\,dx$ & $\int_{-x}^{x} g(x)\,dx$):

Table (3)

Some areas under the curves of

$f_2(x)$ and $g(x)$ from -x to x

x		Area under	
Description	Value	$f_2(x)$	$g(x)$
Point of inflection of g	1.0	0.757383	0.6826
Point of inflection of f_2	1.2867441	0.8992689	0.8030
Point of intersection between f_2&g	1.351882	0.922026	0.8230
Arbitrary point	1.45022	0.95	0.8530
Arbitrary point	1.72692	0.99	0.9164
Arbitrary point	2.0	0.999003	0.9544

The previous table shows that the area under the CFPED curve is larger than that under the standard normal curve for all values of x. This is due to the fact that the CFPED curve is more flatter than the standard normal curve and their peaks are equal.

Figure (2) illustrates most of the previous criteria.

Case (3):

When the points of inflection of the CFPED are taken equal to the points of inflection of the standardized normal distribution i.e. $x = \pm 1$. In this case:

$$\pm \left(\frac{3}{4a}\right)^{\frac{1}{4}} = \pm 1$$

from which

$$a = 0.75 \qquad (3\text{-}28)$$

Hence, the complete 4^{th} power exponential density given by (3-1) will be:

$$f_3(x) \equiv 0.5133522\, e^{-0.75\, x^4} \qquad -\infty < x < \infty \qquad (3\text{-}29)$$

The graphs of f_3 and g are shown in figure (3).

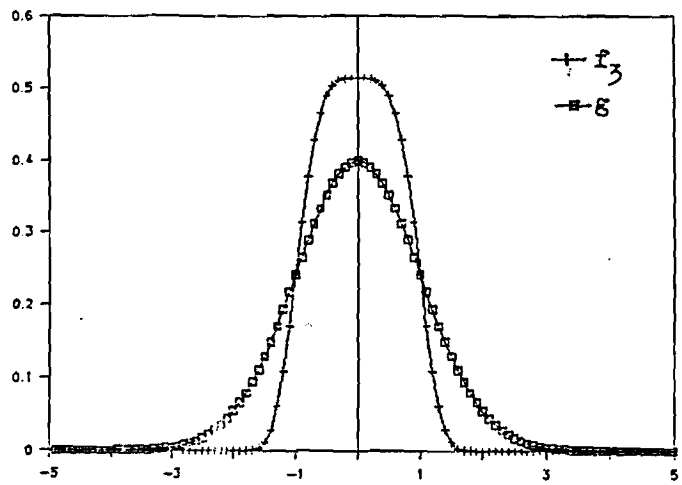

Figure (3)

Substituting the value of a given by (3-28) into (2-6), (3-12) and (3-13), the following phenomena are obtained :

i) The even central moments:

$$\mu_{2r} = \frac{\Gamma(\frac{2r+1}{4})}{(0.75)^{\frac{r}{2}}\ \Gamma(\frac{1}{4})} \qquad r=1,2,3,\ldots \qquad (3\text{-}30)$$

Consequentely. $\mu_2 = 0.5122376$ and $\mu_4 = 0.\bar{3}$

ii) $\lambda = 1.2867438$

Hence

$$f_3(0) = 1.2867438\ g(0) \qquad (3\text{-}31)$$

i.e. The peak of the CFPED is above the peak of the standard normal distribution.

iii) There are two points of intersection between the CFPED curve and the standard normal curve given by:

$x = \pm\ 1.0010553$ (3-32)

iv) The following table gives areas under both the CFPED curve and the standard normal curve at different values of x.(i.e. $\int_{-x}^{x} f_3(x)\,dx$ & $\int_{-x}^{x} g(x)\,dx$):

Table (4)
Some areas under the curves of
$f_3(x)$ and $g(x)$ from -x to x

x		Area under	
Description	Value	$f_3(x)$	$g(x)$
Point of inflection of f_3 & g	1.0	0.899939	0.6826
Arbitrary point	1.12704	0.95	0.7416
Arbitrary point	1.34208	0.99	0.8198
Arbitrary point	2.0	0.999998	0.9544

The previous table shows that the area under the curve of the complete 4^{th} power exponential distribution is greater than that under the standard normal curve for all values of x. This is due to the fact that the curve of the CFPED is more flatter than the normal curve and also, the peak of the CFPED is above that of the standard normal distribution.

From the preceeding cases, it is evident that for variates having pdf more concentrated around the center, the complete 4^{th} power exponential distribution is more suitable.

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